# Second order Taylor polynomial

Let $f : \mathbb{R}^2 \to \mathbb{R}$, such that $f(1, -1) = 0$, $\nabla f(1, -1) = (0, 2)$ and $Hf(1, -1) = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$. Find the second order Taylor polynomial of $f$ at the point $(1, -1)$.

## Solution

The formula for the second order Taylor polynomial of a function is:

$$\begin{aligned}P(x_0, y_0) &= f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)\\ &+ f''_{xx}(x_0, y_0)\frac{(x - x_0)^2}{2!} + f''_{xy}(x_0, y_0)\frac{(x - x_0)(y - y_0)}{2!}\\ &+ f''_{yx}(x_0, y_0)\frac{(x - x_0)(y - y_0)}{2!} + f''_{yy}(x_0, y_0)\frac{(y - y_0)^2}{2!}\end{aligned}$$

Furthermore, we know that:

$$\nabla f(1, -1) = (f'_x(x_0, y_0), f'_y(x_0, y_0)) = (0, 2)$$

$$Hf(1, -1) = \begin{pmatrix} f''_{xx}(x_0, y_0) & f''_{xy}(x_0, y_0) \\ f''_{yx}(x_0, y_0) & f''_{yy}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

Substituting into the function:

$$\begin{aligned}P(x_0, y_0) &= 0 + 0(x - 1) + 2(y + 1)\\ &+ 2\frac{(x - 1)^2}{2} + 1\frac{(x - 1)(y + 1)}{2}\\ &+ 1\frac{(x - 1)(y + 1)}{2} + 4\frac{(y + 1)^2}{2}\end{aligned}$$

$$P(x_0, y_0) = 2(y + 1) + (x - 1)^2 + (x - 1)(y + 1) + 2(y + 1)^2$$